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# Transportation inequalities for stochastic differential equations driven by a fractional Brownian motion

BRUNO SAUSSEREAU

*Laboratoire de Mathématiques de Besançon, CNRS, UMR 6623, 16 Route de Gray, 25030 Besançon cedex, France. E-mail: bruno.saussereau@univ-fcomte.fr*

We establish Talagrand's $T_1$ and $T_2$ inequalities for the law of the solution of a stochastic differential equation driven by a fractional Brownian motion with Hurst parameter $H > 1/2$. We use the $L^2$ metric and the uniform metric on the path space of continuous functions on $[0,T]$. These results are applied to study small-time and large-time asymptotics for the solutions of such equations by means of a Hoeffding-type inequality.

*Keywords:* fractional Brownian motion; fractional calculus; stochastic differential equations; transportation inequalities

## 1. Introduction

Suppose that $B^H = (B^H_t)_{t\in[0,T]}$ is an $m$-dimensional fractional Brownian motion (fBm) with Hurst parameter $H$ defined on a complete filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P})$. By this, we mean that the components $B^{H,j}$, $j = 1,\ldots,m$, are independent centered Gaussian processes with the covariance function

$$R_H(s,t) = \tfrac{1}{2}(t^{2H} + s^{2H} - |t-s|^{2H}).$$

If $H = 1/2$, then $B^H$ is clearly a Brownian motion. Since for any $p \ge 1$, $\mathbb{E}|B^{H,j}_t - B^{H,j}_s|^p = c_p|t-s|^{pH}$, the processes $B^{H,j}$ have $\alpha$-Hölder continuous paths for all $\alpha \in (0,H)$ (see [24] for further information about fBm).

In this article we fix $1/2 < H < 1$ and are interested in the solution $(X_t)_{t\in[0,T]}$ of the stochastic differential equation

$$X^i_t = x^i + \sum_{j=1}^m \int_0^t \sigma^{i,j}(X_s)\,\mathrm{d}B^{H,j}_s + \int_0^t b^i(X_s)\,\mathrm{d}s, \qquad t\in[0,T], \tag{1}$$

$i = 1,\ldots,d$, where $x \in \mathbb{R}^d$ is the initial value of the process $X$.

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Under suitable assumptions on $\sigma$, the processes $\sigma(X)$ and $B^H$ have trajectories which are Hölder continuous of order strictly larger than $1/2$, so we can use the integral introduced by Young in [34]. The stochastic integral in (1) is then a pathwise Riemann–Stieltjes integral. A first result on the existence and uniqueness of a solution of such an equation was obtained in [21] using the notion of $p$-variation. The theory of rough paths introduced by Lyons in [21] was used by Coutin and Qian in order to prove an existence and uniqueness result for the equation 1 (see [6]). The Riemann–Stieltjes integral appearing in equation (1) can be expressed as a Lebesgue integral using a fractional integration by parts formula (see Zähle [35]). Using this formula, Nualart and Răşcanu have established in [25] the existence of a unique solution for a class of general differential equations that includes (1). Regularity (in the sense of Malliavin calculus) and absolute continuity of the law of the random variables $X_t$ have since been investigated in [2, 19, 23, 26].

This work is strongly motivated by the study of the small-time and large-time behaviors of the solution of (1). To the best of our knowledge, little seems to be known on this subject. In [18] the author investigates the ergodicity of the solution when $\sigma$ is constant, as well as the convergence rate toward the stationary solution; see also [22] for infinite-dimensional evolution equations driven by an fBm in an additive way. We will be able to state small-time and large-time asymptotic properties as consequences of stronger properties: the concentration inequalities on the path space of continuous functions.

For several years, the transportation cost-information inequalities and their applications to diffusion processes have been widely studied. In this paper we apply recent results on fractional differential equations in order to obtain Talagrand's inequalities. Let us now consider the kinds of inequalities we will deal with. To measure distances between probability measures, we use transportation distances, also called Wasserstein distances. Let $(E,d)$ be a metric space equipped with a $\sigma$-field $\mathcal{B}$ such that the distance $d$ is $\mathcal{B}\otimes\mathcal{B}$-measurable. Given $p\in[1,+\infty]$ and two probability measures $\mu$ and $\nu$ on $E$, the Wasserstein distance is defined by

$$W_p^d(\mu,\nu)=\inf\left(\iint \mathrm{d}(x,y)^p\,\mathrm{d}\pi(x,y)\right)^{1/p},$$

where the infimum is taken over all the probability measures $\pi$ on $E\times E$ with marginal distributions $\mu$ and $\nu$. The relative entropy of $\nu$ with respect to $\mu$ is defined as

$$\mathbf{H}(\nu/\mu)=\begin{cases}\displaystyle\int \log\frac{\mathrm{d}\nu}{\mathrm{d}\mu}\,\mathrm{d}\nu, & \text{if } \nu\ll\mu,\\ +\infty, & \text{otherwise.}\end{cases}$$

The probability measure $\mu$ satisfies the $L^p$ transportation inequality on $(E,d)$ if there exists a constant $C\geq 0$ such that for any probability measure $\nu$,

$$W_p^d(\mu,\nu)\leq\sqrt{2C\mathbf{H}(\nu/\mu)}.$$

As usual, we write $\mu\in T_p(C)$ for this relation. The properties $T_1(C)$ and $T_2(C)$ are of particular interest. The phenomenon of measure concentration is related to $T_1(C)$ (see the monograph of Ledoux [20]).

The property $T_2(C)$ is stronger than $T_1(C)$ but is not so well characterized. It was first established by Talagrand [30] for the Gaussian measure and generalized in [11] to the

framework of an abstract Wiener space; see [3, 27] for the relationship between $T_2(C)$ and other properties such as the Poincaré inequality and Hamilton–Jacobi equations. The logarithmic Sobolev inequality introduced by Gross [17] plays a particular role in this theory since it implies $T_2(C)$ (see [3, 27, 32]).

With regard to the paths of diffusion processes, the $T_2$ transportation inequality with respect to the Cameron–Martin metric was proven in [8] by means of the Girsanov transform. The authors also provided a direct proof of the $T_1$ transportation inequality with respect to the uniform metric using the Gaussian tail criterion (see Section 6 for more details). Later, in [33], $T_2(C)$ was established with respect to the uniform metric. Finally, Gourcy and Wu [15] established the log-Sobolev inequality for the Brownian motion with drift in the $L^2$ metric instead of the usual Cameron–Martin metric. As a consequence, they derived the $T_2(C)$ property with respect to this metric and a concentration inequality (of correct order for large time) for some functionals of the process. In [31], the $T_2(C)$ property with respect to the $L^2$ metric was established for elliptic diffusions on a Riemannian manifold.

In this paper, we investigate the properties $T_1(C)$ and $T_2(C)$ for the law $\mathbb{P}_x$ of the solution $(X_t)_{0\leq t\leq T}$ of the equation (1) in various situations. We work on the space of continuous functions endowed with the uniform metric or the $L^2$ metric. $T_2(C)$ will hold for a multidimensional equation when $\sigma = I_d$ and $d = m$, and for a one-dimensional equation when the diffusion coefficient $\sigma$ is non-constant. It will also be established with respect to the uniform distance rather than the $L^2$ metric. The use of this second metric will be of particular interest when dealing with large-time asymptotics. The $T_1(C)$ property will be proven for a multidimensional equation with a diffusion matrix $\sigma$ that is only a time-dependent function. In the one-dimensional case, the function $\sigma$ may depend on the space variable. This property is proved with respect to the uniform metric for small-time horizon $T$. This restriction to small time is discussed after Theorem 2 and this result is of great interest when we apply it to small-time asymptotics.

The paper is organized as follows. Section 2 is devoted to the statement of our results. In Section 3, we review the usual consequences of transportation inequalities for large- and small-time behavior. Section 4 contains the estimation of the difference of the solutions of two deterministic differential equations driven by Hölder continuous functions of order greater than $1/2$. The method we develop to prove our main results in Section 5 is the counterpart of the usual case: the Gaussian integrability condition for $T_1(C)$, Girsanov's formula and an explicit control for a specific coupling of two paths of the solution of the stochastic differential equation. In the framework of fractional Brownian motion, this control is new, to the best of our knowledge. In Section 6 we make a quite surprising remark about the link between the constant $C$ in a property $T_1(C)$ and a Gaussian tail. A priori this remark is independent of the rest of this work, but it can be helpful when trying to prove $T_1(C)$ via an exponential moment. Finally, a Fernique-type lemma is proved in the Appendix.

## 2. Main results

We consider a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ on which an $m$-dimensional Brownian motion $(W_t)_{t\in[0,T]}$ is defined. We denote by $\mathcal{F}_t = \sigma(W_s, s \leq t)$ the $\sigma$-field generated by $W$

and completed with respect to $\mathbb{P}$. Finally, $B^H = (B_t^H)_{t\in[0,T]}$ is the $m$-dimensional fBm defined on $(\Omega, \mathcal{F}, \mathbb{P})$ transferred from $W$. This means that $B^H$ can be expressed as

$$B_t^{H,i} = \int_0^t K_H(t,s)\,\mathrm{d}W_s^i, \qquad i = 1,\ldots,m, \tag{2}$$

where the square-integrable deterministic kernel $K_H$ is defined by

$$K_H(t,s) = c_H s^{1/2-H} \int_s^t (u-s)^{H-3/2} u^{H-1/2}\,\mathrm{d}u \tag{3}$$

with $c_H = (\frac{H(2H-1)}{\beta(2-2H,H-1/2)})^{1/2}$ for $s < t$ ($\beta$ denotes the beta function). We set $K_H(t,s) = 0$ if $s \geq t$. The process $B^H$ is $\mathcal{F}_t$-adapted.

We will also need some notation. For $0 < \lambda \leq 1$ and $0 \leq a < b \leq T$, we denote by $C^\lambda(a, b; \mathbb{R}^d)$ the space of $\lambda$-Hölder continuous functions $f : [a,b] \to \mathbb{R}^d$, equipped with the norm

$$\|f\|_\lambda := \|f\|_{a,b,\infty} + \|f\|_{a,b,\lambda},$$

where

$$\|f\|_{a,b,\infty} = \sup_{a\leq r\leq b} |f(r)| \quad \text{and} \quad \|f\|_{a,b,\lambda} = \sup_{a\leq r\leq s\leq b} \frac{|f(s)-f(r)|}{|s-r|^\lambda}.$$

We simply write $C^\lambda(a,b)$ when $d = 1$.

We consider various forms of the stochastic differential equation (1). We begin with the equation on $\mathbb{R}^d$

$$X_t^i = x^i + \int_0^t b^i(X_s)\,\mathrm{d}s + \sum_{j=1}^m \int_0^t \sigma^{i,j}(s)\,\mathrm{d}B_s^{H,j}, \qquad t \in [0,T],\ i = 1,\ldots,d \tag{4}$$

and make the following assumptions on the coefficients:

H1(a) there exists some $L_b$ such that for any $i = 1,\ldots,d$ and any $z, z' \in \mathbb{R}^d$,

$$|b(z) - b(z')| \leq L_b |z - z'|;$$

H1(b) there exists some $\beta > 1 - H$ such that $\sigma \in C^\beta(0,T; \mathbb{R}^{d\times m})$.

It has been proven in [25] that under the above assumptions, there exists a unique adapted stochastic process solution to equation (1) whose trajectories are Hölder continuous of order $H - \epsilon$ for any $\epsilon > 0$.

For this kind of equation, we have the following result.

**Theorem 1.** *Assume that the assumptions* (H1) *are satisfied. Then, for each* $0 < T \leq (2L_b)^{-1} \wedge 1$, *there exists a universal constant* $K$, *independent of the initial point* $x$, *such that the law* $\mathbb{P}_x$ *of the solution of equation (4) satisfies the property* $T_1(K\|\sigma\|_\beta T^{2H})$ *on* $C(0,T;\mathbb{R}^d)$, *the space of* $\mathbb{R}^d$*-valued continuous functions on* $[0,T]$ *equipped with the*

*metric $d_\infty$ defined by*

$$d_\infty(\gamma_1, \gamma_2) = \sup_{0 \le t \le T} |\gamma_1(t) - \gamma_2(t)|.$$

Of course, this result will be useful for small-time asymptotics of the process $X$. In the one-dimensional case, we will be able, via a Lamperti transform, to deduce a result for the nonlinear equation

$$X_t = x + \int_0^t b(X_s)\,\mathrm{d}s + \int_0^t \sigma(X_s)\,\mathrm{d}B_s^H, \tag{5}$$

where the coefficients satisfy:

H2(a) the function $b$ is bounded by $B := \sup_{x \in \mathbb{R}} |b(x)|$ and there exists some $L_b$ such that for any $z, z' \in \mathbb{R}$,

$$|b(z) - b(z')| \le L_b |z - z'|;$$

H2(b) there exist some $\sigma_2 > \sigma_1 > 0$ such that for any $x \in \mathbb{R}$,

$$\sigma_1 \le \sigma(x) \le \sigma_2;$$

H2(c) there exists a constant $L_\sigma$ such that for any $z, z' \in \mathbb{R}$,

$$|\sigma(z) - \sigma(z')| \le L_\sigma |z - z'|.$$

**Theorem 2.** *Assume that the hypotheses (*H2*) are satisfied. There exists a universal constant $K$, independent of the initial point $x$, such that the law $\mathbb{P}_x$ of the solution of equation (5) satisfies the property $T_1(K\sigma_2^2 T^{2H})$ on $C(0,T;\mathbb{R})$, provided that $T \le 1 \wedge \frac{\sigma_1^2}{2\sigma_2(L_b\sigma_2 + L_\sigma B)}$.*

Before stating the $T_2$ inequalities, we will explain why the restriction to small time in the statements of the above theorems is in fact quite natural. Imagine the case where $b = 0$ and $d = m = 1$. The processes $X$ and $B^H$ are then equals. It is known (see ([8], Theorem 2.3) or Section 6) that $T_1(C)$ is then equivalent to the fact that there exists some $\delta > 0$ such that

$$C(\delta) = \mathbb{E}(\exp\{\delta \|B^H - \tilde{B}^H\|_{0,T,\infty}^2\}) < \infty,$$

where $B^H$ and $\tilde{B}^H$ are two independent fractional Brownian motions. For $f, \tilde{f} \in C^\beta(0,T)$ with $f(0) = \tilde{f}(0)$, we have $\|f - \tilde{f}\|_{0,T,\infty} \le T^\beta \|f - \tilde{f}\|_{0,T,\beta}$. Then,

$$C(\delta) \le \mathbb{E}(\exp\{\delta T^{2\beta} \|B^H - \tilde{B}^H\|_{0,T,\beta}^2\})$$

and with (22) from Lemma 8 in the Appendix, the above exponential moment will be finite as soon as $\delta T^{2\beta} \times 128(2T)^{2(H-\beta)} \le 1$, which implies that $T$ must be small.

We now return to the statements concerning $T_2$ transportation inequalities. We consider the solution of the stochastic differential equation (4) and make the following additional stability assumption on the coefficient $b$:

(H3) There exists some $B \in \mathbb{R}$ such that for any $x, y \in \mathbb{R}^d$,

$$\langle x - y, b(x) - b(y) \rangle_{\mathbb{R}^d} \le B|x - y|^2.$$

**Theorem 3.** *We consider $\mathbb{P}_x$, the law of the solution of the stochastic differential equation (4). We assume that* (H1) *and* (H3) *are fulfilled. The probability measure $\mathbb{P}_x$ satisfies $T_2(C)$ on the metric space $C(0, T; \mathbb{R}^d)$ with:*

(a) $C = (2/|B|)HT^{2H-1}(1 \vee e^{(2B+|B|)\times T})\|\sigma\|^2_{0,T,\infty}$ *with the metric $d_\infty$;*
(b) $C = (2/B^2)HT^{2H-1}\|\sigma\|^2_{0,T,\infty} c_{B,T}$ *with*

$$c_{B,T} := \begin{cases} \dfrac{\mathrm{e}^{3BT} - 1}{3}, & \text{if } B > 0, \\ 1 - \mathrm{e}^{BT}, & \text{if } B < 0, \end{cases}$$

*when using the metric*

$$d_2(\gamma_1, \gamma_2) = \left( \int_0^T |\gamma_1(t) - \gamma_2(t)|^2 \, \mathrm{d}t \right)^{1/2}.$$

A result for one-dimensional equations with non-constant diffusion coefficients can be deduced from Theorem 3. We assume that $d = m = 1$ and consider the solution of the stochastic differential equation (5). We make the following assumptions on the coefficients:

H4(a) there exists some $L_b$ such that for any $z, z' \in \mathbb{R}$,

$$|b(z) - b(z')| \le L_b |z - z'|;$$

H4(b) there exist some $\sigma_2 > \sigma_1 > 0$ such that for any $x' \in \mathbb{R}$,

$$\sigma_1 \le \sigma(x) \le \sigma_2;$$

H4(c) $b$ and $\sigma$ are differentiable, and there exists some $B \in \mathbb{R}$ such that for any $x \in \mathbb{R}$,

$$b'(x)\sigma(x) - \sigma'(x)b(x) \le B.$$

**Theorem 4.** *Let $d = m = 1$ and assume that the assumptions* (H4) *hold. The law $\mathbb{P}_x$ of the solution of the stochastic differential equation (5) then satisfies the property $T_2(C)$ on the metric space $C(0, T; \mathbb{R})$ where:*

(a) $C = (2\sigma_1\sigma_2^2/|B|)HT^{2H-1}(1 \vee e^{(2B+|B|)\times T/\sigma_1})$ *with the metric $d_\infty$;*
(b) $C = (2\sigma_1^2\sigma_2^2/B^2)HT^{2H-1}c_{B,T}$ *with*

$$c_{B,T} := \begin{cases} \dfrac{\mathrm{e}^{3BT/\sigma_1} - 1}{3}, & \text{if } B > 0, \\ 1 - \mathrm{e}^{BT/\sigma_1}, & \text{if } B < 0, \end{cases}$$

*when one uses the metric $d_2$.*

We note that (H4) implies (H3) when $d=m=1$ and $\sigma$ is identically equal to 1.

The constants $C$ in the above theorems are sharp, in the sense that when $H=1/2$, we get exactly the same constant as in the inequality (5.5) of [8] with the metric $d_2$. For the $T_1$ inequality, the sharpness will be discussed in the next section, where we will apply the above results to study small-time and large-time asymptotics of the solution of a fractional stochastic differential equation (SDE).

## 3. Small-time and large-time asymptotics of the solution of a fractional SDE

The concentration inequalities on the path space of continuous functions are very well adapted to investigate small- and large-time asymptotics of processes. The link between the concentration inequalities and the $L^1$ transportation inequality is proved in [4]. We recall that a measure $\mu$ on the metric space $(E,d)$ satisfies the property $T_1(C)$ if and only if for any Lipschitzian function $F:(E,d)\to\mathbb{R}$, $F$ is $\mu$-integrable and for all $\lambda\in\mathbb{R}$, we have the Gaussian concentration inequality

$$\int_E \exp\biggl(\lambda\biggl(F-\int_E F\,\mathrm{d}\mu\biggr)\biggr)\,\mathrm{d}\mu \le \exp\biggl(C\|F\|_{\mathrm{Lip}}\frac{\lambda^2}{2}\biggr),$$

where

$$\|F\|_{\mathrm{Lip}} = \sup_{x\neq y}\frac{|F(x)-F(y)|}{d(x,y)}.$$

By Chebyshev's inequality and an optimization argument, we obtain the following Hoeffding-type inequality:

$$\mu\biggl(F-\int_E F\,\mathrm{d}\mu > r\biggr) \le \exp\biggl(-\frac{r^2}{2C\|F\|^2_{\mathrm{Lip}}}\biggr) \qquad \forall r>0. \tag{6}$$

We present Hoeffding-type inequalities for the solution $X$ of (1) on the metric space of continuous functions associated with the metrics $d_\infty$ and $d_2$.

Let $V:R^d\to\mathbb{R}$ be a function such that $\|V\|_{\mathrm{Lip}}\le\alpha$. We consider $F$ and $F_\infty$ defined on $C(0,T;\mathbb{R}^d)$ by

$$F(\gamma) = \frac{1}{T}\int_0^T V(\gamma(t))\,\mathrm{d}t,$$

$$F_\infty(\gamma) = \sup_{t\in[0,T]}|\gamma(t)-\gamma(0)|.$$

The function $F$ is $\alpha$-Lipschitzian with respect to $d_\infty$ and $\alpha/\sqrt{T}$-Lipschitzian with respect to the metric $d_2$. As for $F_\infty$, it is 1-Lipschitzian with respect to the metric $d_\infty$. The following properties are consequences of (6).

### Small-time asymptotics

There exists a constant $C$ (depending only on $H$ and $\sigma$) such that if we assume (H1) (resp., (H2)), then the solution of (4) (resp., (5)) satisfies, for all $r > 0$ and small $T$,

$$\mathbb{P}_x\left(\frac{1}{T}\int_0^T [V(X_t) - \mathbb{E}V(X_t)]\,\mathrm{d}t > r\right) \le \exp\left(-\frac{r^2}{C\alpha^2 T^{2H}}\right), \tag{7}$$

and using (6) with the functional $F_\infty$ yields that there exists some $C$ such that

$$\mathbb{P}_x\Big(\Big[\sup_{t\in[0,T]} |X_t - x| - \mathbb{E}\Big(\sup_{t\in[0,T]} |X_t - x|\Big)\Big] > r\Big) \le \exp\left(-\frac{r^2}{2CT^{2H}}\right). \tag{8}$$

### Large-time asymptotics

In the framework of Theorem 3 (resp., Theorem 4), we assume that (H3) (resp., H4(c)) is satisfied for $B < 0$. The solution of equation (4) (resp., equation (5)) satisfies the following: for any $r > 0$,

$$\mathbb{P}_x\left(\frac{1}{T}\int_0^T [V(X_t) - \mathbb{E}V(X_t)]\,\mathrm{d}t > r\right) \le \exp\left(-\frac{r^2 B^2 T^{2-2H}}{4\alpha^2 H \|\sigma\|^2_{0,T,\infty}(1 - e^{BT})}\right) \tag{9}$$

(resp.,

$$\le \exp\left(-\frac{r^2 B^2 T^{2-2H}}{4\alpha^2 H \sigma_1^2 \sigma_2^2 (1 - e^{BT/\sigma_1})}\right)). \tag{10}$$

***Remark.***

(i) When $H = 1/2$, the inequality (8) gives the correct order when $T \to 0+$ (see [8], Remark 5.12(b)). This justifies that the constants $C$ in the $T_1(C)$ properties established in our work are of correct order and are sharp in some sense.

(ii) The estimates (9) and (10) are well adapted to the study of large-time asymptotics of the solutions of (4) and (5). These estimates are sharp, in the sense that when we put $H = 1/2$ into the formula, we obtain the same Hoeffding-type estimate as given in [8] (see Corollary 5.11).

## 4. Deterministic differential equations driven by rough functions

This section deals with deterministic differential equations driven by Hölder continuous functions. These equations are the ones satisfied by the trajectories of the solution of equation (4). Our aim is to prove an estimate with respect to the metric $d_\infty$ for the difference of two solutions of deterministic differential equations driven by two different Hölder continuous functions. This is clearly the first step if we want to use a Gaussian tail criterion.

Suppose that $f \in C^\lambda(a,b)$ and $g \in C^\mu(a,b)$ with $\lambda + \mu > 1$. From [34], the Riemann–Stieltjes integral $\int_a^b f \,\mathrm{d}g$ exists. In [35], the author provides an explicit expression for the integral $\int_a^b f \,\mathrm{d}g$ in terms of fractional derivatives. Let $\alpha$ be such that $\lambda > \alpha$ and $\beta > 1-\alpha$. Supposing that the following limit exists and is finite, we define $g_{b-}(t) = g(t) - \lim_{\varepsilon\downarrow 0} g(b-\varepsilon)$. The Riemann–Stieltjes integral can then be expressed as

$$\int_a^b f_t \,\mathrm{d}g_t = (-1)^\alpha \int_a^b (D^\alpha_{a+} f)(t)(D^{1-\alpha}_{b-} g_{b-})(t)\,\mathrm{d}t, \tag{11}$$

where

$$D^\alpha_{a+} f(t) = \frac{1}{\Gamma(1-\alpha)}\left(\frac{f(t)}{(t-a)^\alpha} + \alpha \int_a^t \frac{f(t)-f(s)}{(t-s)^{\alpha+1}}\,\mathrm{d}s\right)$$

and

$$D^\alpha_{b-} g_{b-}(t) = \frac{(-1)^\alpha}{\Gamma(1-\alpha)}\left(\frac{g(t)-g(b)}{(b-t)^\alpha} + \alpha \int_t^b \frac{g(t)-g(s)}{(s-t)^{\alpha+1}}\,\mathrm{d}s\right).$$

We refer to [28] for further details on fractional operators. We first state the following useful lemma concerning the estimation of integrals like (11). The proof is identical to the one proposed in [19] and so we only highlight some constants.

**Lemma 5.** *For $0 < \beta < 1$ and $f, g$ in $C^\beta(0,T;\mathbb{R}^d)$, there exists a constant $\kappa$ such that for any $0 \le a < b \le T$,*

$$\left|\int_a^b f_t \,\mathrm{d}g_t\right| \le \frac{\kappa}{\beta - 1/2}\ \|g\|_{0,T,\beta}[\|f\|_{a,b,\infty}(b-a)^\beta + \|f\|_{a,b,\beta}(b-a)^{2\beta}]. \tag{12}$$

**Proof.** We choose $\alpha$ such that $1-\beta < \alpha < 1/2$ and use (11) to write that for all $0 \le s,t \le T$,

$$\left|\int_s^t f_r \,\mathrm{d}g_r\right| \le \int_s^t |D^\alpha_{s+} f_r D^{1-\alpha}_{t-} g_{t-}(r)|\,\mathrm{d}r.$$

We have

$$|D^{1-\alpha}_{t-} g_{t-}(r)| \le \frac{\beta}{(\alpha+\beta-1)\Gamma(\alpha)}\|g\|_{0,T,\beta}|t-r|^{\alpha+\beta-1} \quad \text{and}$$

$$|D^\alpha_{s+} f_r| \le \frac{\|f\|_{s,t,\infty}}{\Gamma(1-\alpha)}(r-s)^{-\alpha} + \frac{\alpha\|f\|_{s,r,\beta}}{(\beta-\alpha)\Gamma(1-\alpha)}(r-s)^{\beta-\alpha}.$$

It follows that

$$\left|\int_s^t f_r \,\mathrm{d}g_r\right| \le \frac{\beta\|f\|_{s,t,\infty}\|g\|_{0,T,\beta}}{(\alpha+\beta-1)\Gamma(\alpha)\Gamma(1-\alpha)}\int_s^t (r-s)^{-\alpha}(t-r)^{\alpha+\beta-1}\,\mathrm{d}r$$
$$+ \frac{\beta\alpha\|f\|_{s,t,\beta}\|g\|_{0,T,\beta}}{(\beta-\alpha)(\alpha+\beta-1)\Gamma(\alpha)\Gamma(1-\alpha)}\int_s^t (r-s)^{\beta-\alpha}(t-r)^{\alpha+\beta-1}\,\mathrm{d}r.$$

We use the change of variables $r=(t-s)\xi+s$ and, recalling that the beta function is defined by $\mathcal{B}(a,b)=\int_0^1(1-\xi)^{a-1}\xi^{b-1}\,\mathrm{d}\xi=\frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$, we get

$$\left|\int_s^t f_r\,\mathrm{d}g_r\right|\le k_{\alpha,\beta}\|g\|_{0,T,\beta}[\|f\|_{s,t,\infty}(t-s)^{\beta}+\|f\|_{s,t,\beta}(t-s)^{2\beta}]$$

with

$$\begin{aligned}k_{\alpha,\beta}&=\frac{\beta\mathcal{B}(\alpha+\beta,1-\alpha)}{(\alpha+\beta-1)\Gamma(\alpha)\Gamma(1-\alpha)}+\frac{\alpha\beta\mathcal{B}(\alpha+\beta,1+\beta-\alpha)}{(\alpha+\beta-1)(\beta-\alpha)\Gamma(\alpha)\Gamma(1-\alpha)}\\&\le\frac{\kappa}{\beta-1/2}:=c_\beta.\end{aligned}$$

The fact that $k_{\alpha,\beta}\le\kappa/(\beta-1/2)$, where $\kappa$ is a universal constant independent of $\alpha$ and $\beta$, is proved in [29]. $\square$

Set $1/2<\beta<1$ and let $g,\tilde{g}\in C^{\beta}(0,T;\mathbb{R}^m)$. We shall work with two deterministic differential equations on $\mathbb{R}^d$:

$$\begin{aligned}x_t^i&=x_0^i+\int_0^t b^i(x_s)\,\mathrm{d}s+\sum_{j=1}^m\int_0^t\sigma^{i,j}(s)\,\mathrm{d}g_s^j,\qquad t\in[0,T],\\\tilde{x}_t^i&=x_0^i+\int_0^t b^i(\tilde{x}_s)\,\mathrm{d}s+\sum_{j=1}^m\int_0^t\sigma^{i,j}(s)\,\mathrm{d}\tilde{g}_s^j,\qquad t\in[0,T],\end{aligned}$$

$i=1,\dots,d$, $x_0\in\mathbb{R}^d$.

It is proved in [25], Theorem 5.1 that if $1-\beta<\alpha<1/2$, then each of the above equations has a unique $(1-\alpha)$-Hölder continuous solution. The estimates on the solution $(x_t)_{t\in[0,T]}$ obtained in [25] were improved in [19], Theorem 3.3. Unfortunately, these estimates are unusable in our context. Nevertheless, since the matrix $\sigma$ does not depend on the solution, our framework is more simple, and we quickly prove the estimate we need in the following proposition.

**Proposition 6.** *Let $g$ and $\tilde{g}$ be Hölder continuous of order $1/2<\beta<1$. Under the assumptions (*H1*), we define $\Delta=(2L_b)^{-1}\wedge 1$. For all $T\le\Delta$, there exists a universal constant $K$ such that*

$$\|x-\tilde{x}\|_{0,T,\infty}\le K\|\sigma\|_{\beta}\|g-\tilde{g}\|_{0,T,\beta}T^{\beta}.$$

**Proof.** We restrict ourselves to the case $d=m=1$ for simplicity. We write

$$x_t-\tilde{x}_t=\int_0^t[b(x_r)-b(\tilde{x}_r)]\,\mathrm{d}r+\int_0^t\sigma(r)\,\mathrm{d}[g_r-\tilde{g}_r].$$

Using (12), we may write

$$|x_t-\tilde{x}_t|\le tL_b\|x-\tilde{x}\|_{0,t,\infty}+c_\beta\|g-\tilde{g}\|_{0,t,\beta}\|\sigma\|_{\beta}[t^{\beta}+t^{2\beta}],$$

where $c_\beta = \kappa/(\beta - 1/2)$, and consequently

$$\|x-\tilde{x}\|_{0,t,\infty} \le tL_b\|x-\tilde{x}\|_{0,t,\infty} + c_\beta\|g-\tilde{g}\|_{0,t,\beta}\|\sigma\|_\beta[t^\beta + t^{2\beta}].$$

Therefore the result is proved when $t \le \Delta$. □

# 5. Proofs of the main results

## 5.1. $T_1(C)$ for paths of SDE's driven by an fBm

To prove Theorem 1, we use a sufficient condition that is present in the proof of [8], Theorem 2.3. This is recalled in the following lemma whose proof is entirely contained in the aforementioned proof.

**Lemma 7.** *Let $\mu$ a probability measure on a metric space $(E,d)$. Let $\xi$ and $\xi'$ be two independent random variables valued in $E$ with law $\mu$ defined on some probability space $(\Omega,\mathcal{F},\mathbb{P})$. If*

$$C := 2\sup_{k\ge 1}\left(\frac{k!\mathbb{E}(d(\xi,\xi'))^{2k}}{(2k)!}\right)^{1/k}$$

*is finite, then $\mu$ satisfies the transportation inequality $T_1(C)$ on $(E,d)$.*

We now turn to the proof of Theorem 1 itself.

**Proof of Theorem 1.** Let $(B_t^H)_{t\in[0,T]}$ and $(\tilde{B}_t^H)_{t\in[0,T]}$ be two independent fractional Brownian motions defined on the filtered probability space $(\Omega,\mathcal{F},(\mathcal{F}_t)_{t\in[0,T]},\mathbb{P})$. We denote by $(X_t)_{t\in[0,T]}$ and $(\tilde{X}_t)_{t\in[0,T]}$ the strong solutions of (4) driven by $B$ and $\tilde{B}$, respectively. The $T_1(C)$ property will be implied by the finiteness of

$$C = 2\sup_{k\ge 1}\left(\frac{k!\mathbb{E}(d_\infty^{2k}(X,\tilde{X}))}{(2k)!}\right)^{1/k}.$$

Let $1/2 < \beta < H < 1$ and $T \le \Delta$. Proposition 6 implies that

$$d_\infty^{2k}(X,\tilde{X}) \le K^{2k}\|\sigma\|_\beta^{2k}\|B^H - \tilde{B}^H\|_{0,T,\beta}^{2k}T^{2k\beta}.$$

In the following, the constant $K$ is universal, but may vary from line to line. Taking expectation and using (23) from Lemma 8, we obtain

$$\begin{aligned} C &\le 2\sup_{k\ge 1}\left(\frac{k!K^{2k}\|\sigma\|_\beta^{2k}\ T^{2k\beta}T^{2k(H-\beta)}(2k)!}{k!(2k)!}\right)^{1/k} \\ &\le K\|\sigma\|_\beta T^{2H}, \end{aligned}$$

and the result is proved. □

**Proof of Theorem 2.** If we set

$$F(y)=\int_0^y \frac{\mathrm{d}z}{\sigma(z)},$$

then we can use the change-of-variables formula [35], Theorem 4.3.1 to obtain that $(X_t)_{t\in[0,T]}$ is the unique solution of

$$X_t = x+\int_0^t b(X_s)\,\mathrm{d}s+\int_0^t \sigma(X_s)\,\mathrm{d}B_s^H, \qquad 0\le t\le T,$$

if and only if the process $(Y_t)_{t\in[0,T]}$ defined by $Y_t=F(X_t)$ is the unique solution of

$$Y_t = F(x)+\int_0^t \frac{b(F^{-1}(Y_s))}{\sigma(F^{-1}(Y_s))}\,\mathrm{d}s+B_t^H, \qquad 0\le t\le T. \tag{13}$$

Our result will follow from the stability of the transportation inequalities under a Lipschitzian map (see [8], Lemma 2.1). We consider the map $\Psi$ from the metric space $(C(0,T),d_\infty)$ into itself defined by $\Psi(\gamma)=F^{-1}\circ\gamma$. We have, for $\gamma_1,\gamma_2\in C(0,T)$,

$$d_\infty(\Psi(\gamma_1)-\Psi(\gamma_2))\le \|\Psi'\|_\infty d_\infty(\gamma_1,\gamma_2)$$

and, clearly, $\Psi'=(F^{-1})'=\sigma$. Thus, the map $\Psi$ is $\alpha$-Lipschitzian with $\alpha=\sigma_2$. If $\mathbb{P}_x^X$ (resp., $\mathbb{P}_{F(x)}^Y$) denotes the law of the process $X$ (resp., $Y$), then

$$\mathbb{P}_x^X=\mathbb{P}_{F(x)}^Y\circ F=\mathbb{P}_{F(x)}^Y\circ\Psi^{-1}.$$

We denote by $L_{\tilde b}$ the Lipschitz constant of the function $\tilde b = b\circ F^{-1}/\sigma\circ F^{-1}$. It is easy to check that

$$L_{\tilde b}\le \frac{\sigma_2}{\sigma_1^2}(L_b\sigma_2+L_\sigma B).$$

By Theorem 1, $\mathbb{P}_{F(x)}^Y\in T_1(KT^{2H})$ for $T\le(2L_{\tilde b})^{-1}\wedge 1$, so we have that $\mathbb{P}_x^X\in T_1(K\sigma_2^2T^{2H})$ for $T\le\tau$ with

$$\tau = 1\wedge\frac{\sigma_1^2}{2\sigma_2(L_b\sigma_2+L_\sigma B)}.$$

□

## 5.2. $T_2(C)$ for paths of SDE's driven by an fBm

In this subsection, we prove Theorems 3 and 4. First, we briefly recall some basic facts about stochastic integration with respect to fBm. We refer to [24] for a more detailed treatment.

*Preliminaries*

Let $\mathcal{H}$ be the Hilbert space defined as the closure of $\mathcal{E}$ (the set of step functions on $[0,T]$ with values in $\mathbb{R}^m$) with respect to the scalar product

$$\langle (\mathbf{1}_{[0,t_1]},\ldots,\mathbf{1}_{[0,t_m]}),(\mathbf{1}_{[0,s_1]},\ldots,\mathbf{1}_{[0,s_m]})\rangle_{\mathcal{H}} = \sum_{i=1}^m R_H(t_i,s_i).$$

The mapping $(\mathbf{1}_{[0,t_1]},\ldots,\mathbf{1}_{[0,t_m]}) \mapsto \sum_{i=1}^m B_{t_i}^{H,i}$ is extended to an isometry between $\mathcal{H}$ and the Gaussian space $H_1(B^H)$ associated with $B^H$. We denote this isometry by $\varphi \mapsto B(\varphi)$. Using the kernel $K$ defined in (3), we introduce the operator $\mathcal{K}_H^*:\mathcal{H}\to L^2(0,T;\mathbb{R}^m)$:

$$(\mathcal{K}_H^*\varphi)(s) = \int_s^T \varphi(r)\frac{\partial K_H}{\partial r}(r,s)\,\mathrm{d}r. \tag{14}$$

We have $\mathcal{K}_H^*((\mathbf{1}_{[0,t_1]},\ldots,\mathbf{1}_{[0,t_m]})) = (K_H(t_1,\cdot),\ldots,K_H(t_m,\cdot))$ and, for $\varphi,\psi\in\mathcal{E}$,

$$\langle\varphi,\psi\rangle_{\mathcal{H}} = \langle\mathcal{K}_H^*\varphi,\mathcal{K}_H^*\psi\rangle_{L^2(0,T;\mathbb{R}^m)} = \mathbb{E}(B^H(\varphi)B^H(\psi)).$$

$\mathcal{K}_H^*$ then provides an isometry between the Hilbert space $\mathcal{H}$ and a closed subspace of $L^2(0,T;\mathbb{R}^m)$.

We have already mentioned the transfer principle (see (2)) when $B^H$ is written as an integral of the underlying Brownian motion $W$. More precisely, the transfer principle means that for any $\varphi\in\mathcal{H}$, $B^H(\varphi) = W(\mathcal{K}_H^*\varphi)$.

We define $\mathcal{K}_H : L^2(0,T;\mathbb{R}^m) \to \mathcal{H}_H := \mathcal{K}_H(L^2(0,T;\mathbb{R}^m))$, the operator defined by $\mathcal{K}_H h = (\mathcal{K}_H h^1,\ldots,\mathcal{K}_H h^m)$ with

$$(\mathcal{K}_H h^i)(t) := \int_0^t K_H(t,s)h^i(s)\,\mathrm{d}s, \qquad i=1,\ldots,m.$$

We will use of the following property [7], Lemma 3.2: for $h\in L^2(0,T;\mathbb{R}^m)$,

$$|(\mathcal{K}_H h)(t) - (\mathcal{K}_H h)(s)| \le c|t-s|^H \|h\|_{L^2(0,T;\mathbb{R}^m)}. \tag{15}$$

Using Fubini's theorem and the fact that $\frac{\partial K_H}{\partial u}(u,s) = c_H(\frac{u}{s})^{H-1/2}(u-s)^{H-3/2}$, we obtain that if $f\in C^\lambda(0,T)$ with $\lambda + H > 1$ and $\rho\in L^2(0,T)$, then it holds that

$$\int_0^T f(r)\,\mathrm{d}(\mathcal{K}_H\rho)_r = \int_0^T f(r)\left(\int_0^r \frac{\partial K_H}{\partial r}(r,t)\rho(t)\,\mathrm{d}t\right)\mathrm{d}r. \tag{16}$$

The integral on the left-hand side of (16) is a Riemann–Stieltjes integral for Hölder functions (see Section 4).

Finally, if $\varphi,\psi\in L^2(0,T;\mathbb{R}^m)$, then the scalar product on $\mathcal{H}$ has the integral form

$$\langle\varphi,\psi\rangle = H(2H-1)\int_0^T\int_0^T |s-t|^{2H-2}\langle\varphi(s),\psi(t)\rangle_{\mathbb{R}^m}\,\mathrm{d}s\,\mathrm{d}t$$

and, consequently, for $\varphi\in L^2(0,T;\mathbb{R}^m)$, we have

$$\|\varphi\|_{\mathcal{H}}^2 \le 2HT^{2H-1}\|\varphi\|_{L^2(0,T;\mathbb{R}^m)}^2. \tag{17}$$

**Proof of Theorem 3.** We recall that a classical $m$-dimensional Brownian motion $(W_t)_{t\in[0,T]}$ is defined on $(\Omega,\mathcal{F},\mathbb{P})$ and $B^H=(B^H_t)_{t\in[0,T]}$ is an $m$-dimensional fBm defined on $(\Omega,\mathcal{F},\mathbb{P})$ transferred from $W$. Let $\mathbb{Q}$ be a probability measure on $C(0,T;\mathbb{R}^d)$ such that $\mathbb{Q}\ll\mathbb{P}_x$. We can assume that $\mathbf{H}(\mathbb{Q}|\mathbb{P}_x)<\infty$, otherwise there is nothing to prove.

The first part of the proof follows the arguments of [8]. The idea is to express the finiteness of the entropy by means of the energy of the drift arising from the Girsanov transform of a well-chosen probability measure. This method also appears in [10] and was well known for a long time. The relationship between the finite entropy condition and the finite energy condition on the Girsanov drift appeared in [12, 13] for the first time (to the best of our knowledge) in the particular case of Brownian motion with drift.

We consider

$$\tilde{\mathbb{Q}}=\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}_x}(X)\mathbb{P}.$$

Clearly, $\tilde{\mathbb{Q}}$ is a probability measure on $(\Omega,\mathcal{F})$ and

$$\begin{aligned}\mathbf{H}(\tilde{\mathbb{Q}}|\mathbb{P})&=\int_\Omega\ln\left(\frac{\mathrm{d}\tilde{\mathbb{Q}}}{\mathrm{d}\mathbb{P}}\right)\mathrm{d}\tilde{\mathbb{Q}}\\&=\int_\Omega\ln\left(\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}_x}(X)\right)\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}_x}(X)\,\mathrm{d}\mathbb{P}\\&=\int_{C(0,T;\mathbb{R}^d)}\ln\left(\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}_x}\right)\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}_x}\,\mathrm{d}\mathbb{P}_x=\mathbf{H}(\mathbb{Q}|\mathbb{P}_x).\end{aligned}$$

Following [8], there exists a predictable process $\rho=(\rho^1(t),\dots,\rho^m(t))_{0\le t\le T}$ such that

$$\mathbf{H}(\mathbb{Q}|\mathbb{P}_x)=\mathbf{H}(\tilde{\mathbb{Q}}|\mathbb{P})=\frac{1}{2}\mathbb{E}_{\tilde{\mathbb{Q}}}\int_0^T|\rho(t)|^2\,\mathrm{d}t$$

and, by Girsanov's theorem, the process $(\tilde{B}_t)_{t\in[0,T]}$ defined by

$$\tilde{B}_t=W_t-\int_0^t\rho(s)\,\mathrm{d}s$$

is a Brownian motion under $\tilde{\mathbb{Q}}$ and is associated (thanks to the transfer principle) with the $\tilde{\mathbb{Q}}$-fractional Brownian motion $(\tilde{B}^H)_{t\in[0,T]}$ defined by

$$\tilde{B}^H_t=\int_0^t K_H(t,s)\,\mathrm{d}\tilde{B}_s=\int_0^t K_H(t,s)\,\mathrm{d}W_s-(\mathcal{K}_H\rho)(t)=B^H_t-(\mathcal{K}_H\rho)(t).$$

Consequently, under $\tilde{\mathbb{Q}}$, $X$ verifies

$$\begin{cases}\mathrm{d}X_t=b(X_t)\,\mathrm{d}t+\sigma(t)\,\mathrm{d}\tilde{B}^H_t+\sigma(t)\,\mathrm{d}(\mathcal{K}_H\rho)(t),\\X_0=x.\end{cases}\tag{18}$$

We now consider the solution $Y$ (under $\tilde{\mathbb{Q}}$) of the following equation:

$$\begin{cases} \mathrm{d}Y_t = b(Y_t)\,\mathrm{d}t + \sigma(t)\,\mathrm{d}\tilde{B}_t^H \\ Y_0 = x. \end{cases} \tag{19}$$

Under $\tilde{\mathbb{Q}}$, the law of the process $(Y_t)_{t\in[0,T]}$ is exactly $\mathbb{P}_x$. Then, $(X,Y)$ under $\tilde{\mathbb{Q}}$ is a coupling of $(\mathbb{Q},\mathbb{P}_x)$ and it follows that

$$[W_2^{d_2}(\mathbb{Q},\mathbb{P}_x)]^2 \le \mathbb{E}_{\tilde{\mathbb{Q}}}(|d_2(X,Y)|^2) = \mathbb{E}_{\tilde{\mathbb{Q}}}\biggl(\int_0^T |X_t - Y_t|^2\,\mathrm{d}t\biggr),$$
$$[W_2^{d_\infty}(\mathbb{Q},\mathbb{P}_x)]^2 \le \mathbb{E}_{\tilde{\mathbb{Q}}}(|d_\infty(X,Y)|^2) = \mathbb{E}_{\tilde{\mathbb{Q}}}\Bigl(\sup_{0\le t\le T}|X_t - Y_t|^2\Bigr).$$

We now estimate the distance on $C(0,T;\mathbb{R}^m)$ between $X$ and $Y$ with respect to the distances $d_2$ and $d_\infty$. We note that equations (18) and (19) can be considered as pathwise integral equations driven by $\beta$-Hölder functions with $\beta < H$. Indeed, the Hölder regularity is straightforward for the driving function $\tilde{B}$ since it is a fractional Brownian motion under $\tilde{\mathbb{Q}}$ (and so it has almost surely $\beta$-Hölder trajectories for any $\beta < H$). Moreover, since $\int_0^T |\rho(s)|^2\,\mathrm{d}s < +\infty$ almost surely, $\mathcal{K}_H\rho \in C^H(0,T)$ almost surely by (15).

We write

$$X_t - Y_t = \int_0^t (b(X_s) - b(Y_s))\,\mathrm{d}s + \int_0^t \sigma(s)\,\mathrm{d}(\mathcal{K}_H\rho)(s).$$

We use the change of variables formula for a $\beta$-Hölder continuous function (see [35], Theorem 4.3.1) and the stability assumption (H2) to obtain

$$\begin{aligned} |X_t - Y_t|^2 &= 2\sum_{i=1}^d\sum_{j=1}^m \int_0^t (X_s^i - Y_s^i)\sigma^{i,j}(s)\,\mathrm{d}(\mathcal{K}_H\rho)^j(s) \\ &\quad + 2\int_0^t \langle X_s - Y_s, b(X_s) - b(Y_s)\rangle_{\mathbb{R}^d}\,\mathrm{d}s \\ &\le 2\sum_{i=1}^d\sum_{j=1}^m \int_0^t (X_s^i - Y_s^i)\sigma^{i,j}(s)\,\mathrm{d}(\mathcal{K}_H\rho^j)(s) + 2B\int_0^t |X_s - Y_s|^2\,\mathrm{d}s. \end{aligned} \tag{20}$$

Since $X - Y \in C^\beta(0,T;\mathbb{R}^d)$ and $\rho \in L^2(0,T;\mathbb{R}^m)$, we use (14) and (16) to obtain

$$\begin{aligned} &\int_0^t (X_s^i - Y_s^i)\sigma^{i,j}(s)\,\mathrm{d}(\mathcal{K}_H\rho^j)(s) \\ &\quad = \int_0^t (X_s^i - Y_s^i)\sigma^{i,j}(s)\biggl(\int_0^s \frac{\partial K_H}{\partial s}(s,r)\rho^j(r)\,\mathrm{d}r\biggr)\mathrm{d}s \end{aligned}$$

$$= \int_0^t \left( \int_r^t (X_s^i - Y_s^i) \sigma^{i,j}(s) \frac{\partial K_H}{\partial s}(s,r)\, \mathrm{d}s \right) \rho^j(r)\, \mathrm{d}r$$

$$= \int_0^t \mathcal{K}_H^*((X^i - Y^i)\sigma^{i,j} \mathbf{1}_{[0,t]})(r) \rho^j(r)\, \mathrm{d}r.$$

We denote by $\sigma^*$ the transpose matrix of $\sigma$ and we use the inequality (17) to obtain

$$\begin{aligned}
&2 \sum_{i=1}^d \sum_{j=1}^m \int_0^t (X_s^i - Y_s^i) \sigma^{i,j}(s)\, \mathrm{d}(\mathcal{K}_H \rho)^j(s) \\
&\quad = 2 \int_0^t \langle \mathcal{K}_H^*(\sigma^*(X-Y)\mathbf{1}_{[0,t]})(r), \rho(r) \rangle_{\mathbb{R}^m}\, \mathrm{d}r \\
&\quad \le 2 \|\mathcal{K}_H^*(\sigma^*(X-Y)\mathbf{1}_{[0,t]})\|_{L^2(0,T)} \|\rho\|_{L^2(0,t)} \\
&\quad \le 2 \|\sigma^*(X-Y)\mathbf{1}_{[0,t]}\|_{\mathcal{H}} \|\rho\|_{L^2(0,t)} \\
&\quad \le 2(2H)^{1/2} T^{H-1/2} \|\sigma^*(X-Y)\mathbf{1}_{[0,t]}\|_{L^2(0,T)} \|\rho\|_{L^2(0,t)} \\
&\quad \le 2(2H)^{1/2} T^{H-1/2} \|\sigma\|_{0,T,\infty} \|X-Y\|_{L^2(0,t)} \|\rho\|_{L^2(0,t)}.
\end{aligned}$$

We report this estimate in (20), and using the inequality $4\epsilon ab \le 4\epsilon^2 a^2 + b^2$ with $\epsilon = (HT^{2H-1}\|\sigma\|^2_{0,T,\infty}/(2|B|))^{1/2}$, we obtain

$$\begin{aligned}
|X_t - Y_t|^2 &\le 2(2H)^{1/2} T^{H-1/2} \|\sigma\|_{0,T,\infty} \|X-Y\|_{L^2(0,t)} \|\rho\|_{L^2(0,t)} \\
&\quad + 2B \int_0^t |X_s - Y_s|^2\, \mathrm{d}s \\
&\le (2/|B|) HT^{2H-1} \|\sigma\|^2_{0,T,\infty} \int_0^t |\rho(s)|^2\, \mathrm{d}s \\
&\quad + (2B + |B|) \int_0^t |X_s - Y_s|^2\, \mathrm{d}s.
\end{aligned}$$

Gronwall's lemma implies that for any $t > 0$,

$$|X_t - Y_t|^2 \le (2/|B|) HT^{2H-1} \|\sigma\|^2_{0,T,\infty} \int_0^t \mathrm{e}^{(2B+|B|)\times(t-s)} |\rho(s)|^2\, \mathrm{d}s.$$

Hence, we may write that

$$d_\infty^2(X,Y) \le (2H/|B|) T^{2H-1} \|\sigma\|^2_{0,T,\infty} (1 \vee \mathrm{e}^{(2B+|B|)\times T}) \int_0^T |\rho(s)|^2\, \mathrm{d}s$$

and

$$[W_2^{d_\infty}(\mathbb{Q}, \mathbb{P}_x)]^2 \le 2C_{T,H} \mathbf{H}(\mathbb{Q}|\mathbb{P}_x)$$

with $C_{T,H} = 2HT^{2H-1}(1 \vee \mathrm{e}^{(2B+|B|)\times T}) \|\sigma\|^2_{0,T,\infty}/|B|$.

Analogously for the metric $d_2$, we have

$$
\begin{aligned}
[W_2^{d_2}(\mathbb{Q},\mathbb{P}_x)]^2 &\le \mathbb{E}_{\tilde{\mathbb{Q}}}\int_0^T |X_t-Y_t|^2\,\mathrm{d}t\\
&\le (2/|B|)HT^{2H-1}\|\sigma\|_{0,T,\infty}^2\\
&\quad\times \mathbb{E}_{\tilde{\mathbb{Q}}}\int_0^T |\rho(s)|^2\left(\int_s^T \mathrm{e}^{(2B+|B|)\times(t-s)}\,\mathrm{d}t\right)\mathrm{d}s.
\end{aligned}
$$

Since

$$
\int_s^T \mathrm{e}^{(2B+|B|)\times(t-s)}\,\mathrm{d}t \le \begin{cases} \dfrac{\mathrm{e}^{3BT}-1}{3B}, & \text{if } B>0,\\ -\dfrac{1-\mathrm{e}^{BT}}{B}, & \text{if } B<0,\end{cases}
$$

we define

$$
c_{B,T} := \begin{cases} \dfrac{\mathrm{e}^{3BT}-1}{3}, & \text{if } B>0,\\ 1-\mathrm{e}^{BT}, & \text{if } B<0\end{cases}
$$

and it follows that

$$
\begin{aligned}
[W_2^{d_2}(\mathbb{Q},\mathbb{P}_x)]^2 &\le 4(H/B^2)T^{2H-1}\|\sigma\|_{0,T,\infty}^2 c_{B,T}\left(\frac{1}{2}\mathbb{E}_{\tilde{\mathbb{Q}}}\int_0^T |\rho(s)|^2\,\mathrm{d}s\right)\\
&\le 2C_{T,H}\mathbf{H}(\mathbb{Q}|\mathbb{P}_x)
\end{aligned}
$$

with $C_{T,H}=(2/B^2)HT^{2H-1}\|\sigma\|_{0,T,\infty}^2 c_{B,T}$. □

**Proof of Theorem 4.** We use the same change-of-variables as in the proof of Theorem 2 and consider the map $\Psi$ from the metric space $(C(0,T),d_2)$ into itself defined by $\Psi(\gamma)=F^{-1}\circ\gamma$. We have, for $\gamma_1,\gamma_2\in C(0,T)$,

$$
\begin{aligned}
d_2(\Psi(\gamma_1)-\Psi(\gamma_2)) &= \left(\int_0^T |\Psi(\gamma_1(s))-\Psi(\gamma_2(s))|^2\,\mathrm{d}s\right)^{1/2}\\
&\le \|\Psi'\|_\infty d_2(\gamma_1,\gamma_2),
\end{aligned}
$$

thus the map $\Psi$ is $\sigma_2$-Lipschitzian. If $\mathbb{P}_x^X$ (resp., $\mathbb{P}_{F(x)}^Y$) denotes the law of the process $X$ (resp., $Y$), then

$$
\mathbb{P}_x^X=\mathbb{P}_{F(x)}^Y\circ F=\mathbb{P}_{F(x)}^Y\circ\Psi^{-1}.
$$

Since $\mathbb{P}_{F(x)}^Y\in T_2(C)$, we have that $\mathbb{P}_x^X\in T_2(\sigma_2^2C)$. It remains to prove that the stability assumption (H3) is true for the function $\tilde{b}=b\circ F^{-1}/\sigma\circ F^{-1}$. Writing $\tilde{b}'=(b'\circ F^{-1}\sigma\circ F^{-1}-b\circ F^{-1}\sigma'\circ F^{-1})/\sigma\circ F^{-1}$, it easy to see that under the assumptions (H4), we

have

$$\left(x-y, \frac{(b\circ F^{-1})(x)}{(\sigma\circ F^{-1})(x)} - \frac{(b\circ F^{-1})(y)}{(\sigma\circ F^{-1})(y)}\right) \le \frac{B}{\sigma_1}|x-y|^2.$$

We can then apply Theorem 3 to equation (13) and thus the result (b) is proved. A similar reasoning is true for the metric $d_\infty$. $\square$

## 6. A remark on the link between the exponential moment and $T_1(C)$

It has been proven in [4, 5, 8] that $\mu \in T_1(C)$ if and only if we have, for some $\delta > 0$, the Gaussian tail

$$\int_E \int_E \mathrm{e}^{\delta d^2(x,y)} \mu(\mathrm{d}x)\mu(\mathrm{d}y) < +\infty.$$

The link between the constant $C$ and the exponential moment is described in the following remark.

***Remark.*** Let $\mu$ a probability measure on a metric space $(E,d)$. Assume that there exists some $\delta > 0$ such that the following Gaussian tail holds:

$$C(\delta) := \int_E \int_E \mathrm{e}^{\delta d^2(x,y)} \mu(\mathrm{d}x)\mu(\mathrm{d}y) < +\infty.$$

Then, $\mu$ satisfies the transportation inequality $T_1(C)$ on $(E,d)$. In [8], the authors have linked $C$ with the above exponential moment in the following way:

$$C \le \frac{2}{\delta} \sup_{k\ge 1} \left( \frac{(k!)^2}{(2k)!} \int_E \int_E \mathrm{e}^{\delta d^2(x,y)} \mu(\mathrm{d}x)\mu(\mathrm{d}y) \right)^{1/k}. \tag{21}$$

By an optimization argument, the supremum in the formula (21) is achieved for $k=1$ and consequently $C \le C(\delta)/\delta$.

In [5] (see also [16], page 69), the authors have proven that the constant $C$ is in fact controlled by a better constant, but it is not tractable to study short-time and long-time asymptotic behavior.

In our context, if we use the above remark and the exponential estimate (22) from Lemma 8, then we can easily prove that the law $\mathbb{P}_x$ of the solution of equation (4) satisfies the property $T_1(C)$ with $C = K\|\sigma\|_\beta T^{2H-\varepsilon}$ for small time $T$ and a small $\varepsilon > 0$. Nevertheless, the power of $T$ is not the correct order when one applies this result to small-time asymptotics.

We believe that it remains an interesting open problem to give a simple link between the exponential moment and the constant $C$ in $T_1(C)$.

**Proof of the estimate $C \le C(\delta)/\delta$.** We use an optimization argument involving the gamma function $\Gamma$. We denote, for $x \ge 1$,

$$\Phi(x) = \exp\left(\frac{1}{x}\ln\left(C(\delta)\frac{\Gamma^2(x+1)}{\Gamma(2x+1)}\right)\right).$$

We remark that the right-hand side of (21) is equal to $(2/\delta)\Phi(k)$. Our result will then be a consequence of $\sup_{x\ge 1}\Phi(x) = \Phi(1) = C(\delta)/2$. We denote by $\Psi$ the function $(\ln\Gamma)' = \Gamma'/\Gamma$ (usually called the *digamma function*). We write $\Phi'(x) = h(x)\Phi(x)/x^2$, where the function $h$ is defined for $x \ge 1$ by

$$h(x) = -\ln\left(C(\delta)\frac{\Gamma^2(x+1)}{\Gamma(2x+1)}\right) + 2x(\Psi(x+1) - \Psi(2x+1)).$$

Obviously, $\Phi'$ and $h$ have the same sign. Since $\Psi'(x) = \sum_{k=0}^{\infty}\frac{1}{(x+k)^2}$ (see [1], page 13), we deduce that

$$\begin{aligned}
&\Psi'(x+1) - 2\Psi'(2x+1)\\
&\quad= \sum_{k=0}^{\infty}\frac{1}{(x+1+k)^2} - \frac{1}{2(x+(k+1)/2)^2}\\
&\quad= \frac{1}{2}\sum_{k=0}^{\infty}\frac{1}{(x+1+k)^2} + \frac{1}{2}\sum_{k=0}^{\infty}\frac{1}{(x+1+k)^2} - \frac{1}{(x+(k+1)/2)^2}\\
&\quad= \frac{1}{2}\left\{\sum_{k=0}^{\infty}\frac{1}{(x+1+k)^2} + \sum_{j=0}^{\infty} - \frac{1}{(x+(2j+1)/2)^2}\right\}\\
&\quad= \frac{1}{2}\left\{\sum_{k=0}^{\infty}\frac{1}{(x+1+k)^2} + \sum_{k=0}^{\infty} - \frac{1}{(x+k+1/2)^2}\right\}\\
&\quad= \frac{1}{2}\{\Psi'(x+1) - \Psi'(x+1/2)\}.
\end{aligned}$$

Since $\Psi''(x) = -2\sum_{k=0}^{\infty}(x+k)^{-3}$, $\Psi'$ is a decreasing function and then

$$\Psi'(x+1) - 2\Psi'(2x+1) \le 0.$$

This yields $h'(x) = 2x(\Psi'(x+1) - 2\Psi'(2x+1)) \le 0$. So, for any $x \ge 1$,

$$h(x) \le h(1) = -\ln(C(\delta)\Gamma^2(2)/\Gamma(3)) + 2(\Psi(2) - \Psi(3)).$$

In [1], the following identity is stated for $n \ge 1$:

$$\Psi(x+n) = \sum_{k=0}^{n-1}\frac{1}{x+k} + \Psi(x),$$

so $\Psi(2)-\Psi(3)=-1/2$. Finally, $h(1)=-\ln(C(\delta)/2)-1\leq 0$ because $C(\delta)\geq 1$. Thus, $h(x)\leq 0$ for any $x\geq 1$, and $\Phi$ is decreasing. Its maximum is achieved for $x=1$. $\square$

## Appendix: Fernique-type lemma

**Lemma 8.** *Let $T>0$, $1/2<\beta<H<1$. Then, for any $\alpha<1/(128(2T)^{2(H-\beta)})$,*

$$\mathbb{E}[\exp(\alpha\|B^H\|^2_{0,T,\beta})]\leq(1-128\alpha(2T)^{2(H-\beta)})^{-1/2}. \tag{22}$$

*Moreover, we have the following moment estimate for any $k\geq 1$:*

$$\mathbb{E}(\|B^H\|^{2k}_{0,T,\beta})\leq 32^k(2T)^{2k(H-\beta)}\frac{(2k)!}{k!}. \tag{23}$$

**Proof.** First, we prove that

$$|B^{H,i}_t-B^{H,i}_s|\leq\xi_\beta|t-s|^\beta,\qquad i=1,\dots,m, \tag{24}$$

where $\xi_\beta$ is a positive random variable such that

$$E(\xi_\beta^{2p})\leq 32^p(2T)^{2p(H-\beta)}\frac{(2p)!}{p!}. \tag{25}$$

Although the proofs of (24) and (25) are classical, we include them for the convenience of the reader. With $\psi(u)=u^{2/(H-\beta)}$ and $p(u)=u^H$ in Lemma 1.1 of [14], the Garsia–Rodemich–Rumsey inequality reads as follows:

$$|B^{H,i}_t-B^{H,i}_s|\leq 8\int_0^{|t-s|}\left(\frac{4\Delta}{u^2}\right)^{(H-\beta)/2}Hu^{H-1}\,\mathrm{d}u,$$

where the random variable $\Delta$ is

$$\Delta=\int_0^T\int_0^T\frac{|B^{H,i}_t-B^{H,i}_s|^{2/(H-\beta)}}{|t-s|^{2H/(H-\beta)}}\,\mathrm{d}t\,\mathrm{d}s.$$

We have

$$\begin{aligned}|B^{H,i}_t-B^{H,i}_s|&\leq 8(4\Delta)^{(H-\beta)/2}\int_0^{|t-s|}Hu^{\beta-1}\,\mathrm{d}u\leq 8(4\Delta)^{(H-\beta)/2}\;\frac{H}{\beta}|t-s|^\beta\\&\leq 8(4\Delta)^{(H-\beta)/2}|t-s|^\beta.\end{aligned}$$

We let $\xi_\beta=8(4\Delta)^{(H-\beta)/2}$ and for $p\geq 1/(H-\beta)$, we have

$$\mathbb{E}\xi_\beta^{2p}\leq 8^{2p}4^{p(H-\beta)}\mathbb{E}\left(\int_0^T\int_0^T\frac{|B^{H,i}_t-B^{H,i}_s|^{2/(H-\beta)}}{|t-s|^{2H/(H-\beta)}}\,\mathrm{d}t\,\mathrm{d}s\right)^{p(H-\beta)}$$

$$\begin{aligned}&\le 8^{2p}(2T)^{2p(H-\beta)}\int_0^T\int_0^T \frac{\mathbb{E}|B_t^{H,i}-B_s^{H,i}|^{2p}}{|t-s|^{2pH}}\frac{\mathrm{d}t\,\mathrm{d}s}{T^2}\\ &\le 8^{2p}(2T)^{2p(H-\beta)}\frac{(2p)!}{2^p p!}\le 32^p(2T)^{2p(H-\beta)}\frac{(2p)!}{p!}.\end{aligned}$$

Thus, (24) and (25) are proved. What remains to be shown can be tediously deduced from [9], Theorem 1.3.2. We can also make the following direct computations. Using (24) and (25), we have

$$\begin{aligned}\mathbb{E}(\exp(\alpha\|B^H\|_\beta^2)) \le \mathbb{E}(\exp(\alpha\xi_\beta^2)) &\le \mathbb{E}\left(\sum_{p=0}^{\infty}\frac{\alpha^p\xi_\beta^{2p}}{p!}\right)\\ &\le \sum_{p=0}^{\infty}(32\alpha)^p(2T)^{2p(H-\beta)}\frac{(2p)!}{(p!)^2}\\ &\le (1-128\alpha(2T)^{2(H-\beta)})^{-1/2},\end{aligned}$$

where we have used the identity $\sum_{p=0}^{\infty}a^p\frac{(2p)!}{(p!)^2}=(1-4a)^{-1/2}$ for $a<1/4$. Thus, the lemma is proved. $\square$

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